Department of Physics, University of Florida — Spring 2014-15
PHZ 7608 Minicourse

Problem Set 1: Due February 18

1. In post-Newtonian theory, there appears a "superpotential" X defined by

$$X(t,\boldsymbol{x}) = G\int \rho(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|d^3x' \,.$$

Show that $\nabla^2 X = 2U$, and that

$$\begin{aligned}\frac{\partial^2}{\partial t^2}X(t,\boldsymbol{x}) &= -G\int \rho'\frac{d\boldsymbol{v}'}{dt}\cdot\frac{(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x' \\ &\quad + G\int \frac{\rho'}{|\boldsymbol{x}-\boldsymbol{x}'|}\left\{v'^2 - \frac{[\boldsymbol{v}'\cdot(\boldsymbol{x}-\boldsymbol{x}')]^2}{|\boldsymbol{x}-\boldsymbol{x}'|^2}\right\}d^3x' \,.\end{aligned}$$

2. For $\ell = 2,\ 3$ and 4, show explicitly that $n'^{\langle L\rangle}n^{\langle L\rangle} = [\ell!/(2\ell-1)!!]P_\ell(\mu)$, where $\mu := \boldsymbol{n'}\cdot\boldsymbol{n}$.

3. Suppose that the solar system is filled with a uniform distribution of dark matter with constant mass density ρ. Taking this distribution into account, calculate the modified gravitational potential of the Sun, and find the perturbing force $\boldsymbol{f}$ acting on a planetary orbit. Find the relation between orbital period P and semi-major axis a for a circular orbit, and calculate the secular changes in the planet's orbital elements. Place a bound on ρ using suitable solar-system data.

4. Consider a spherical body on an inclined, circular orbit about an axisymmetric body of radius R and even multipole moments J_ℓ, with $\ell = 2, 4, 6$, and so on. To first order in perturbation theory, calculate the secular changes in the relevant orbital elements. In particular, show that:

   (a) the inclination is constant, that is, $\Delta\iota = 0$;

   (b) the line of nodes changes by an amount

$$\Delta\Omega = -3\pi\cos\iota\sum_{\ell=2}^{\infty} J_\ell\left(\frac{R}{p}\right)^\ell C_\ell,$$

   where $C_2 = 1$, $C_4 = -\frac{5}{2}(1-\frac{7}{4}\sin^2\iota)$, and $C_6 = \frac{35}{8}(1-\frac{9}{2}\sin^2\iota+\frac{33}{8}\sin^4\iota)$.

5. Show that $g_{\alpha\beta} = \sqrt{-\mathfrak{g}}\,\mathfrak{g}_{\alpha\beta}$, where $\mathfrak{g}_{\alpha\beta}$ is the matrix inverse to $\mathfrak{g}^{\alpha\beta}$, and $\mathfrak{g} = \det[\mathfrak{g}^{\alpha\beta}] = g$. If we define $\mathfrak{g}^{\alpha\beta} := \eta^{\alpha\beta} - h^{\alpha\beta}$, and $h^{\alpha\beta}$ is of order G, show that

$$\begin{aligned}(-g) &= 1 - h + \frac{1}{2}h^2 - \frac{1}{2}h^{\mu\nu}h_{\mu\nu} + O(G^3)\,, \\ g_{\alpha\beta} &= \eta_{\alpha\beta} + h_{\alpha\beta} - \frac{1}{2}h\eta_{\alpha\beta} + h_{\alpha\mu}h^\mu{}_\beta - \frac{1}{2}hh_{\alpha\beta} \\ &\quad + \left(\frac{1}{8}h^2 - \frac{1}{4}h^{\mu\nu}h_{\mu\nu}\right)\eta_{\alpha\beta} + O(G^3)\,,\end{aligned}$$

where indices on $h^{\alpha\beta}$ are lowered and contracted with the Minkowski metric.

6. Consider the Schwarzschild metric in harmonic coordinates, given by

$$\begin{aligned} g_{00} &= -\frac{1 - R/2r_{\rm h}}{1 + R/2r_{\rm h}}, \\ g_{jk} &= \left(\frac{1 + R/2r_{\rm h}}{1 - R/2r_{\rm h}}\right) n_j n_k + \big(1 + R/2r_{\rm h}\big)^2 \big(\delta_{jk} - n_j n_k\big), \end{aligned}$$

where $n^j := x^j/r_{\rm h}$ is a radial unit vector, whose index is lowered with the Euclidean metric δ_{jk}, so that $n_j := \delta_{jk} n^k$. Show explicitly that

$$\begin{aligned} \mathfrak{g}^{00} &= -\frac{(1 + R/2r)^3}{1 - R/2r}, \\ \mathfrak{g}^{jk} &= \delta^{jk} - \left(\frac{R}{2r}\right)^2 n^j n^k, \end{aligned}$$

where $R := 2GM/c^2$, and verify that the harmonic gauge condition $\partial_\beta \mathfrak{g}^{\alpha\beta} = 0$ is satisfied.

7. Verify the identities

$$\begin{aligned} \tau^{0j} &= \partial_0\big(\tau^{00} x^j\big) + \partial_k\big(\tau^{0k} x^j\big), \\ \tau^{jk} &= \frac{1}{2}\partial_{00}\big(\tau^{00} x^j x^k\big) + \frac{1}{2}\partial_p\big(2\tau^{p(j} x^{k)} - \partial_q \tau^{pq} x^j x^k\big), \\ \tau^{0j} x^k &= \frac{1}{2}\partial_0\big(\tau^{00} x^j x^k\big) + \tau^{0[j} x^{k]} + \partial_p\big(\tau^{0p} x^j x^k\big), \\ \tau^{jk} x^n &= \frac{1}{2}\partial_0\big(2\tau^{0(j} x^{k)} x^n - \tau^{0n} x^j x^k\big) \\ &\quad + \frac{1}{2}\partial_p\big(2\tau^{p(j} x^{k)} x^n - \tau^{np} x^j x^k\big). \end{aligned} \tag{0.1}$$

Using these identities verify that the near-zone expansion

$$h^{00}_{\mathscr{N}}(t, \boldsymbol{x}) = \frac{4G}{c^4} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell! c^\ell} \left(\frac{\partial}{\partial t}\right)^\ell \int_{\mathscr{M}} \tau^{00}(t, \boldsymbol{x'}) |\boldsymbol{x} - \boldsymbol{x'}|^{\ell - 1}\, d^3x'$$

takes the form

$$\begin{aligned} h^{00}_{\mathscr{N}} = \frac{4G}{c^2} \bigg\{ & \int_{\mathscr{M}} \frac{c^{-2}\tau^{00}}{|\boldsymbol{x} - \boldsymbol{x'}|}\, d^3x' + \frac{1}{2c^2} \frac{\partial^2}{\partial t^2} \int_{\mathscr{M}} c^{-2}\tau^{00} |\boldsymbol{x} - \boldsymbol{x'}|\, d^3x' \\ & - \frac{1}{6c^3} \overset{(3)}{\mathcal{I}}{}^{kk}(t) + \frac{1}{24c^4} \frac{\partial^4}{\partial t^4} \int_{\mathscr{M}} c^{-2}\tau^{00} |\boldsymbol{x} - \boldsymbol{x'}|^3\, d^3x' \\ & - \frac{1}{120c^5} \Big[(4x^k x^l + 2r^2 \delta^{kl}) \overset{(5)}{\mathcal{I}}{}^{kl}(t) - 4x^k \overset{(5)}{\mathcal{I}}{}^{kll}(t) + \overset{(5)}{\mathcal{I}}{}^{kkll}(t) \Big] \\ & + O(c^{-6}) \bigg\} + h^{00}[\partial \mathscr{M}], \end{aligned}$$

modulo surface integrals denoted by $h^{00}[\partial\mathscr{M}]$, where $\mathcal{I}^L(t) := \int_{\mathscr{N}} \tau^{00} x^L d^3x$ and the symbol (n) on top of $\mathcal{I}$ denotes the number of time derivatives.